BITS Pilani
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MECHANICS OF SOLIDS (ME F211)

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Chapter-7

Stresses due to Bending

Stresses due to Bending

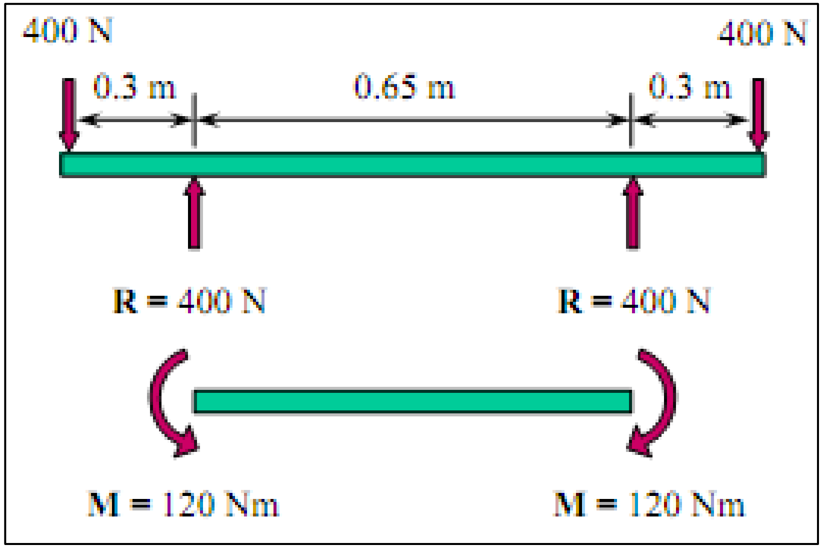
Objectives

- Discuss – Stresses and strains associated with Shear Force and Bending Moment.
- To develop the relationship between stresses, Bending moments, young's modulus, Moment of inertia, strains, Radius of curvature and so on.
- Analyze the stress distributions inside the slender member or beams (beams are transversely loaded slender members)

Stresses due to Bending

Pure Bending

Prismatic members subjected to equal and opposite couples acting in the same longitudinal plane



Stresses due to Bending

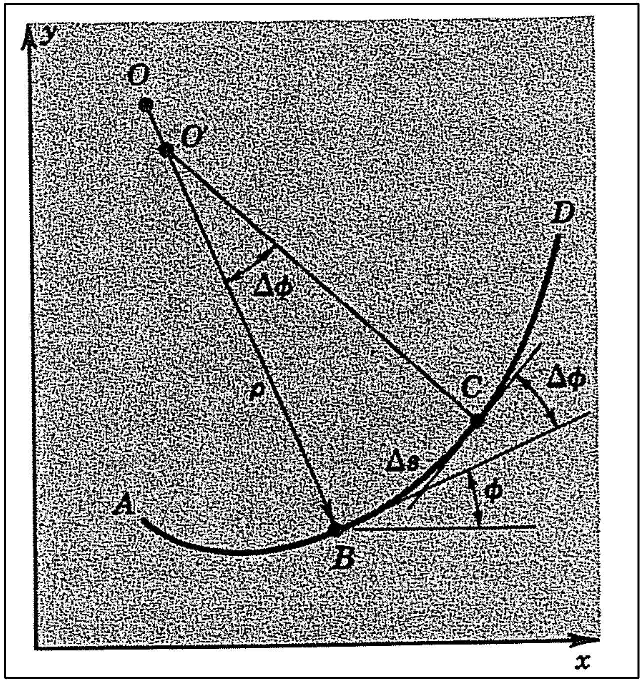
Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

Curvature: the rate of change of the slope angle of the curve with respect to distance along the curve.

- The normals to the curve at *B* and *C* intersect in the point *O'*.
- The change in the slope angle between *B* and *C* is $\Delta\phi$.
- When $\Delta\phi$ is small, the arc $\Delta s = O'B\,\Delta\phi$.

$\Delta s \to 0$, the curvature at point B is defined as

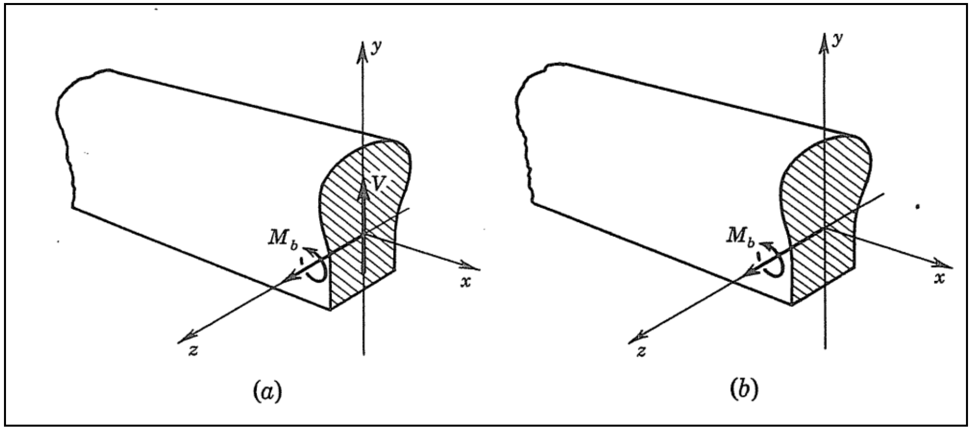
$$\frac{d\phi}{ds} = \lim_{\Delta s \to 0} \frac{\Delta\phi}{\Delta s} = \lim_{\Delta s \to 0} \frac{1}{O'B} = \frac{1}{\rho}$$



Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

Assumptions in the simple theory of bending

- The beam is initially straight
- Cross section of beam is symmetrical about plane of loading and it is constant.
- Beam transmits constant bending moment i.e. case of pure bending .



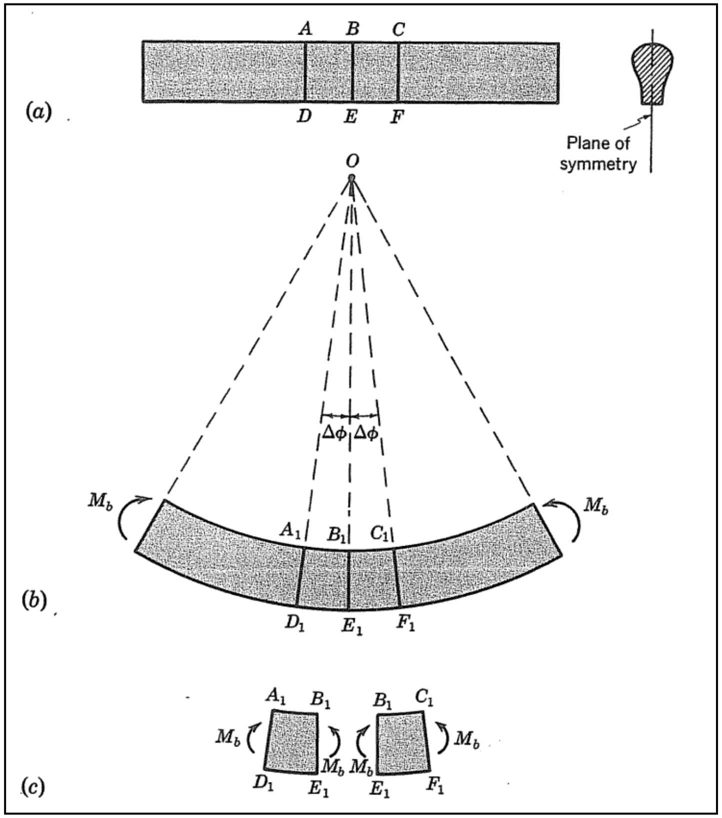
(a) In general, both shear force and bending moments are transmitted
(b) In pure bending is no shear force, and a constant bending moment is transmitted

Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

Assumptions in the simple theory of bending

- Plane transverse sections, normal to the axis of the beam remain plane and normal to the axis of the beam after bending

  i.e there is no distortion of the cross section.



Stresses due to Bending

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Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

Assumptions in the simple theory of bending

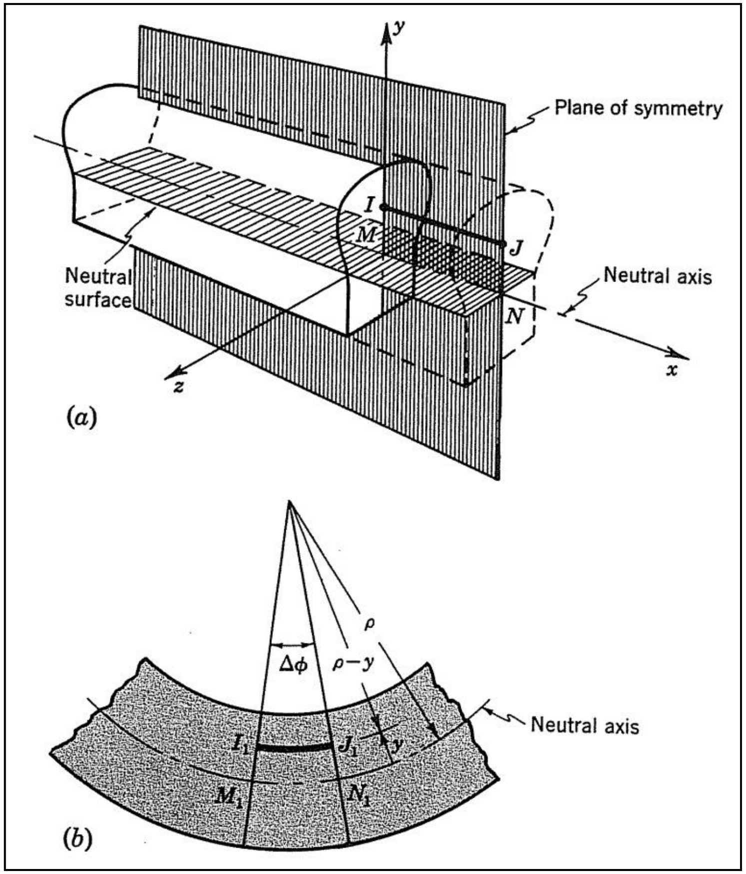
- The material of the beam is homogeneous and isotropic and it obeys Hooke's law at all points.
- Every layer of the material is free to expand or contract longitudinally and laterally under stress and do not exert pressure upon each other.
- *E* is same in tension and compression

Stresses due to Bending

Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

After applying constant bending moment

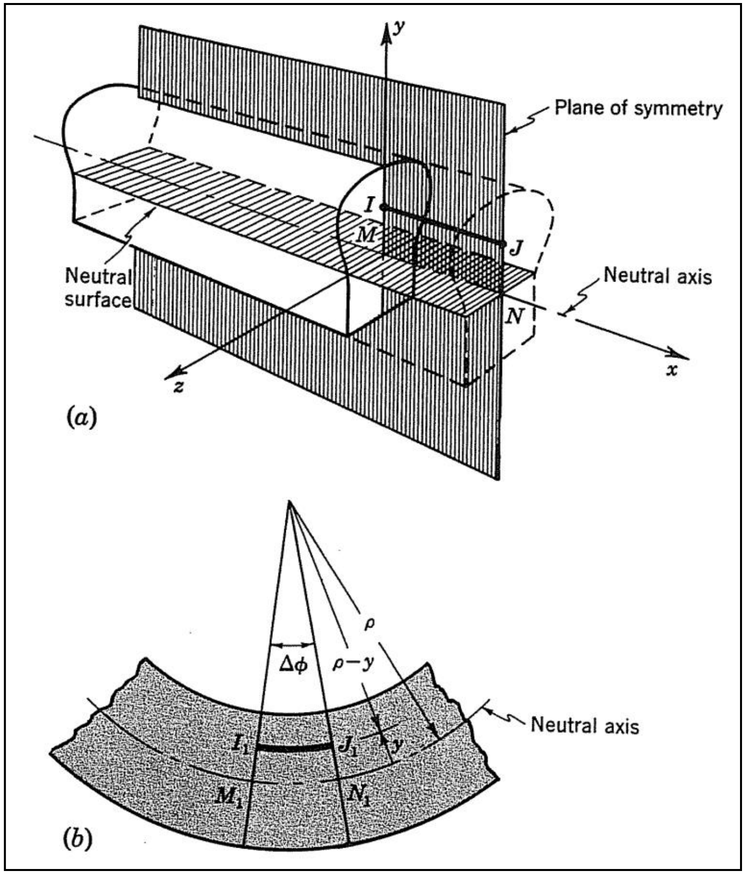
- ❑ Some lines are shortened & some elongated
- ❑ There is one line in the pane of symmetry which has not changed in length, called Neutral Axis.
- ❑ Yet the precise location of Neutral Axis is unknown.
- ❑ Setup coordinate system in such a way that *x* axis coincides with neutral axis.
- ❑ The *xy*-plane is the plane of symmetry and the *xz*-plane is called the neutral surface



Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

After applying constant bending moment

- *IJ* and *MN* are separated by distance *y* in the unreformed beam.
- They are deformed into concentric circular arcs I_1J_1 and M_1N_1.
- We assume that the difference between their radii of curvature is still *y*.
- Let ρ be the radius of curvature of the deformed neutral axis M_1N_1.
- The radius of curvature of I_1J_1 is then $\rho - y$



Stresses due to Bending

Geometry of Deformation of a Symmetrical Beam Subjected to Pure Bending

Since $IJ = MN = M_1N_1$ from the definition of neutral axis, the strain of I_1J_1 is

$$\varepsilon_x = \frac{I_1J_1 - IJ}{IJ} = \frac{I_1J_1 - M_1N_1}{M_1N_1}$$

$$M_1N_1 = \rho\Delta\phi \quad \text{and} \quad I_1J_1 = (\rho - y)\Delta\phi$$

$$\varepsilon_x = -\frac{y}{\rho} = -\frac{d\phi}{ds}y$$

- The strain varies linearly with y
- Symmetry arguments which require plane section to remain plane that

$$\gamma_{xy} = \gamma_{xz} = 0$$

Stresses due to Bending

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Stresses obtained from Stress-Strain Relations

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right] = -\frac{y}{\rho}$$

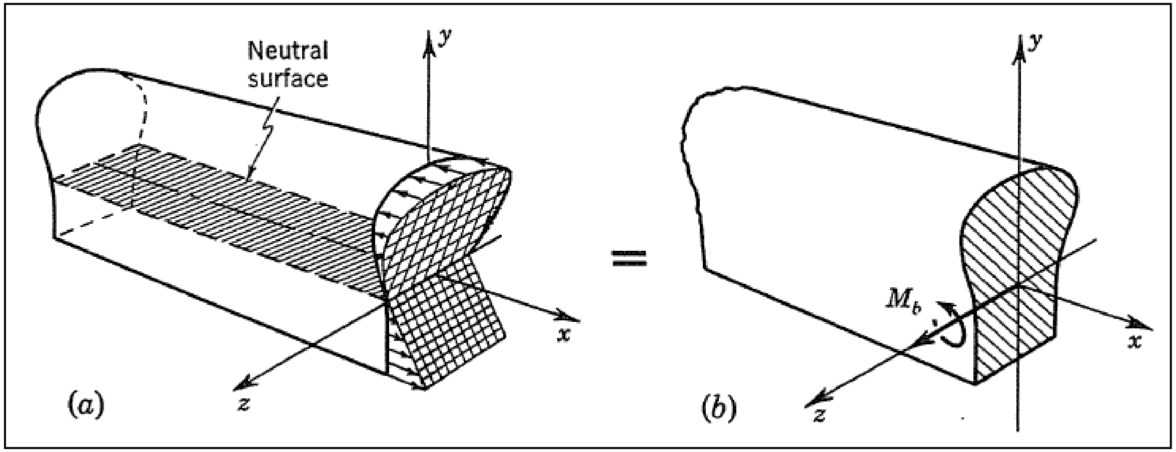
$$\gamma_{xy} = \frac{\tau_{xy}}{G} = 0 \quad \Rightarrow \quad \tau_{xy} = 0$$

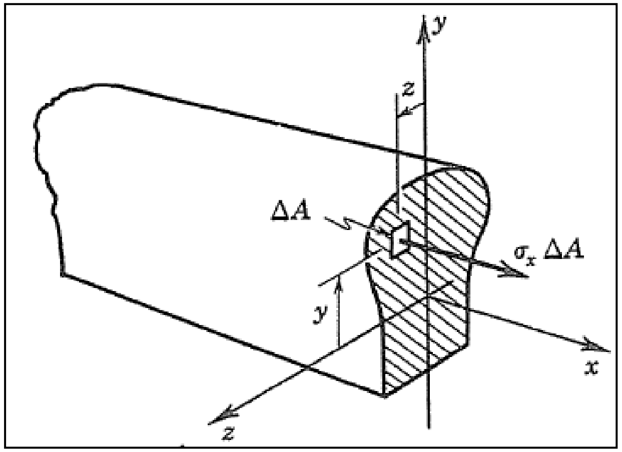
$$\gamma_{xz} = \frac{\tau_{xz}}{G} = 0 \quad \Rightarrow \quad \tau_{xz} = 0$$

Thus the shear-stress components τ_{xy} and τ_{xz} vanish in pure bending

Stresses due to Bending

Equilibrium Requirements





The resultant of the stress distribution in pure bending must be the bending moment M_b.

Force acting on an elemental area ΔA of the beam.

$$\sum F_x = \int_A \sigma_x dA = 0 \quad ; \quad \sum M_y = \int_A z\sigma_x dA = 0 \quad ; \quad \sum M_z = -\int_A y\sigma_x dA = M_b$$

Stresses due to Bending

Stress and Deformation in Symmetrical Elastic Beam Subjected to Pure Bending

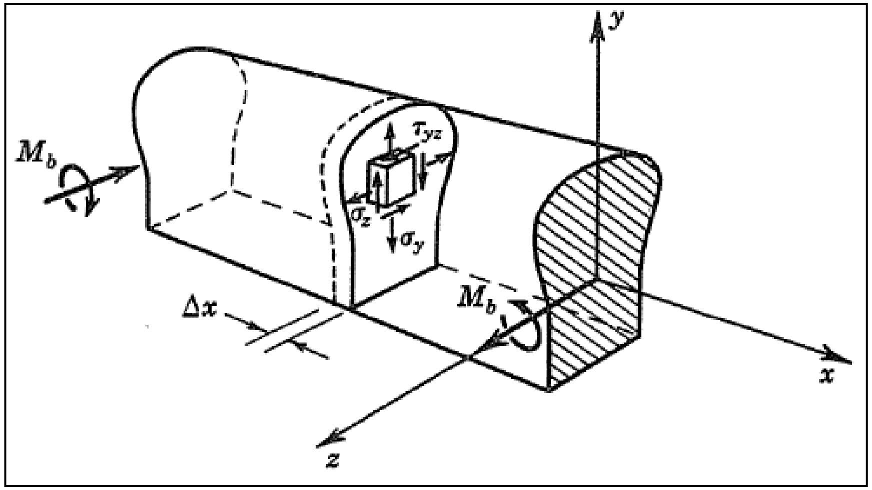
$$\sigma_y = \sigma_z = \tau_{yz} = 0$$

From stress strain relations

$$\sigma_x = -E\frac{y}{\rho} = -E\frac{d\phi}{ds}y$$

From equilibrium equations

$$\sum F_x = \int_A \sigma_x dA = -\int_A E\frac{y}{\rho}dA = -\frac{E}{\rho}\int_A y dA = 0$$



The transverse stresses σ_y, σ_z and τ_{yz} are assumed to be zero

Above equation implies that the neutral surface must pass through the centroid of the cross-sectional area.

Stress and Deformation in Symmetrical Elastic Beam Subjected to Pure Bending

From equilibrium equations

$$\sum M_y = \int_A z\sigma_x dA = -\int_A E\frac{y}{\rho} zdA = -\frac{E}{\rho}\int_A yzdA = 0$$

Above equation satisfy because of symmetry of the cross section with respective to the *xy* plane.

$$\sum M_z = -\int_A y\sigma_x dA = \int_A yE\frac{y}{\rho} zdA = \frac{E}{\rho}\int_A y^2 dA = M_b$$

The integral in the above equation is known as second moment of area or Moment of Inertia of the area about the neutral axis.

Stresses due to Bending

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Stress and Deformation in Symmetrical Elastic Beam Subjected to Pure Bending

- Moment of inertia can be calculated once the specific shape of cross-section is known.
- Since this moment of inertia is about *z* axis, we denote it by I_{zz}.

$$I_{zz} = \int_A y^2 dA$$

Substituting I_{zz} in previous equation, we obtain expression for curvature as a function of bending moment

$$\frac{d\phi}{ds} = \frac{1}{\rho} = \frac{M_b}{EI_{zz}}$$

When bending moment is positive, the curvature is positive, that is, concave upward.

Stresses due to Bending

Stress and Deformation in Symmetrical Elastic Beam Subjected to Pure Bending

Stress and strain in terms of applied bending moment

$$\varepsilon_x = -\frac{y}{\rho} = -\frac{M_b y}{EI_{zz}}$$

$$\sigma_x = -E\frac{y}{\rho} = -\frac{M_b y}{I_{zz}}$$

- The stress and strain distribution is linear.
- *y* is distance measured from neutral axis.
- The fibers on top surface of the beam are in compression while the fibers on the bottom surface are in tension in case of positive bending moment

Flexural Formula

From curvature and stress expression

$$\frac{E}{\rho} = \frac{\sigma_{bt}}{y} = -\frac{\sigma_{bc}}{y} = \frac{M_b}{I_{zz}}$$

- σ_{bt} and σ_{bc} are bending stresses in tension and compression respectively.
- The distance *y* should be taken accordingly.

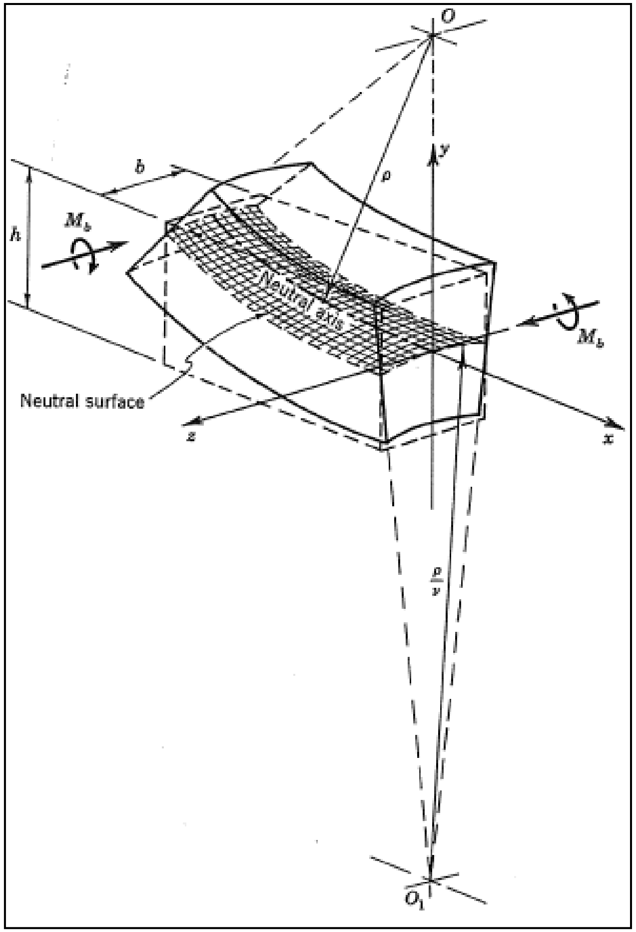
Stresses due to Bending

Stress and Deformation in Symmetrical Elastic Beam Subjected to Pure Bending

Transverse strain components.

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = \nu \frac{M_b y}{EI_{zz}}$$

$$\gamma_{yz} = 0$$

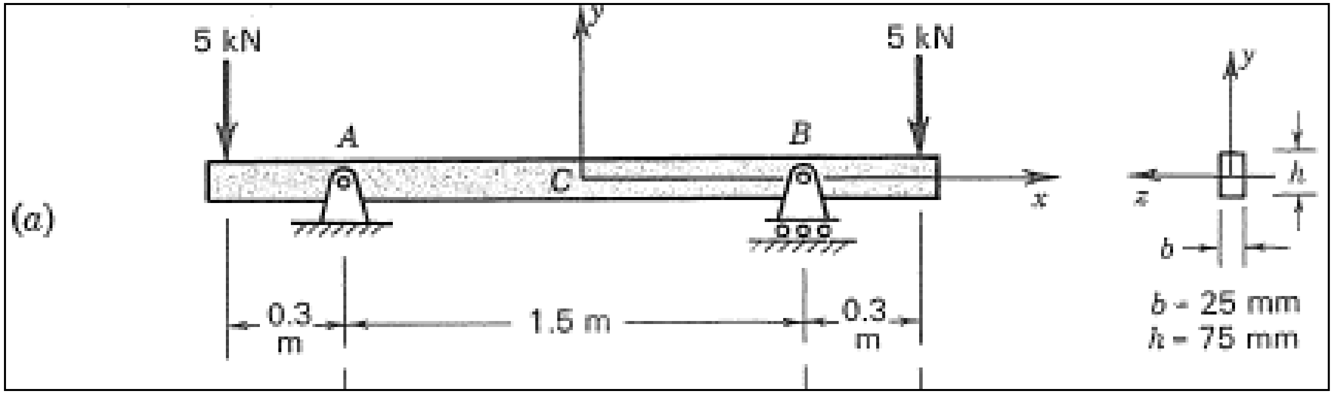


- Compressed region expand laterally
- Tensile region contract laterally
- Neutral surface actually has a double curvature, one is in *xy* plane and another is in *yz* plane.
- The later Curvature is called **anticlastic Curvature**

Stresses due to Bending

Example

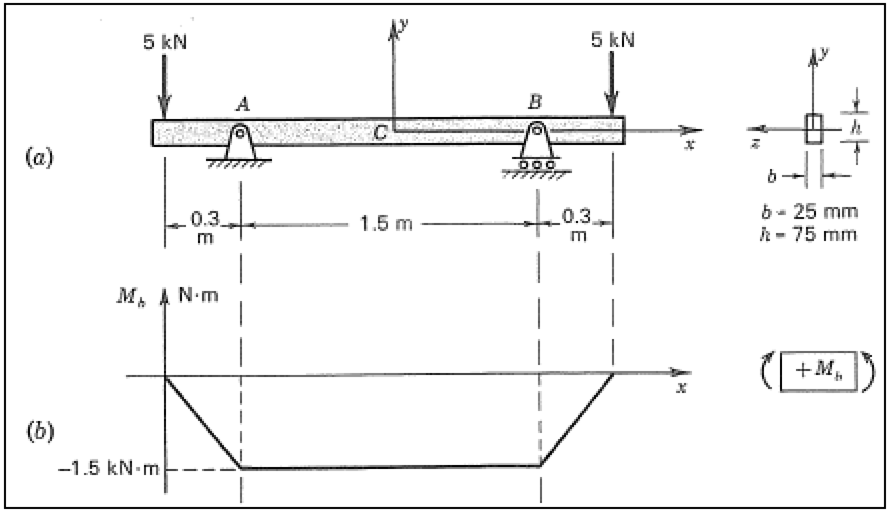
A steel beam 25 mm wide and 75 mm deep is pinned to supports at points *A* & *B*, where the support *B* is on rollers and free to move horizontally. When the ends of the beam are loaded with 5kN loads, we wish to find the maximum bending stress at the mid span of the beam.



Stresses due to Bending

Solution

- Maximum bending moment in the beam is -1.5kNm (Ref. Ch.3)
- Moment of inertia for rectangular section is



$$I_{zz}=\frac{1}{12}bh^3=\frac{1}{12}\times 25\times(75)^3=878.91\times 10^3 mm^4$$

- From Flexural formula

$$\frac{\sigma_x}{y}=-\frac{M_b}{I_{zz}}$$

- y is distance from neutral axis to extreme fiber i.e. 37.5mm.
- Since the figure is symmetric about *z* axis, tensile and compressive bending stresses will be same

Stresses due to Bending

Solution

$$\sigma_x = -\frac{M_b}{I_{zz}} y = -\frac{\left(-1.5\times 10^3 \times 10^3\right)}{878.91\times 10^3} \times 37.5 = 64MPa$$

Stresses due to Bending

Section Modulus

Section modulus is used to compare c/s of the beam symmetric @ *y* and *z* axes.

e.g. Square c/s Vs Rectangle c/s

Circular c/s Vs Rectangle c/s

Circular c/s Vs Elliptical c/s and so on

From flexural formula

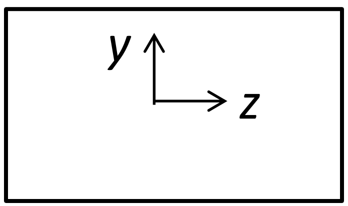
$$\sigma_b = \frac{M_b}{I_{ZZ}} y = \frac{M_b}{Z}$$

where 'Z' is section modulus in mm^3

$$Z = \frac{I_{zz}}{y}$$

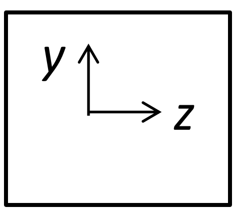
Stresses due to Bending

Section Modulus



$b \times h$

$$I_{zz} = \frac{bh^3}{12}; \qquad y = \frac{h}{2}; \qquad Z = \frac{bh^2}{6}$$

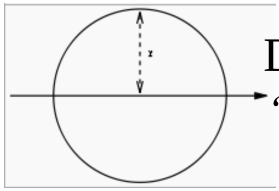


$a \times a$

$$I_{zz} = \frac{a^4}{12}; \quad y = \frac{a}{2}; \quad Z = \frac{a^3}{6}$$

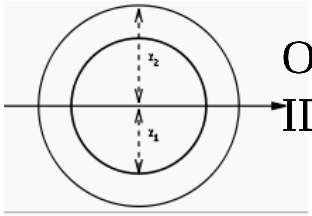
Stresses due to Bending

Section Modulus



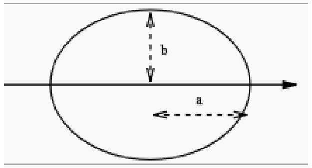
Diameter *'d'*

$$I_{zz} = \frac{\pi d^4}{64}; \quad y = \frac{d}{2}; \quad Z = \frac{\pi d^3}{32}$$



OD *'D'*
ID *'d'*

$$I_{zz} = \frac{\pi (D^4 - d^4)}{64}; \quad y = \frac{D}{2}; \quad Z = \frac{\pi (D^4 - d^4)}{32D}$$



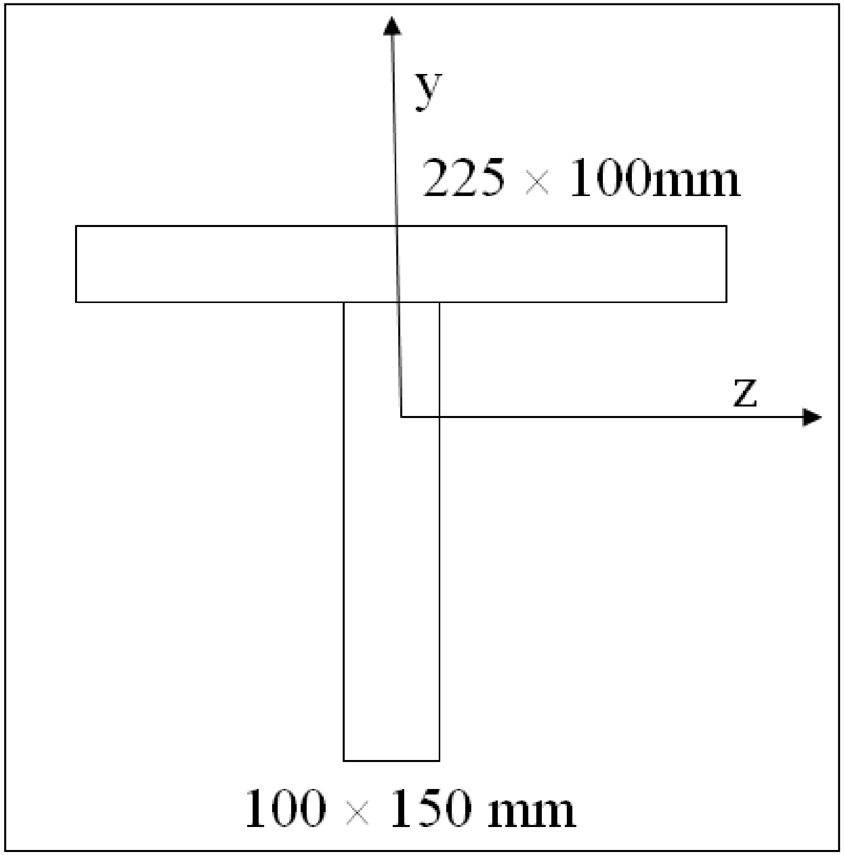
Major axis *'2a'*
Minor axis *'2b'*

$$I_{zz} = \frac{\pi a b^3}{4}; \quad y = b; \quad Z = \frac{\pi a b^2}{4}$$

Stresses due to Bending

Problem:

Calculate the moment of inertia for the beam cross section illustrated.



Stresses due to Bending

Solution:

$$\bar{y}_1 = 200mm \qquad \bar{y}_2 = 75mm$$

$$\bar{y} = 150mm$$

$$(I_{ZZ})_1 = \frac{1}{12}\times 225\times(100)^3 + 225\times 100\times(50)^2$$

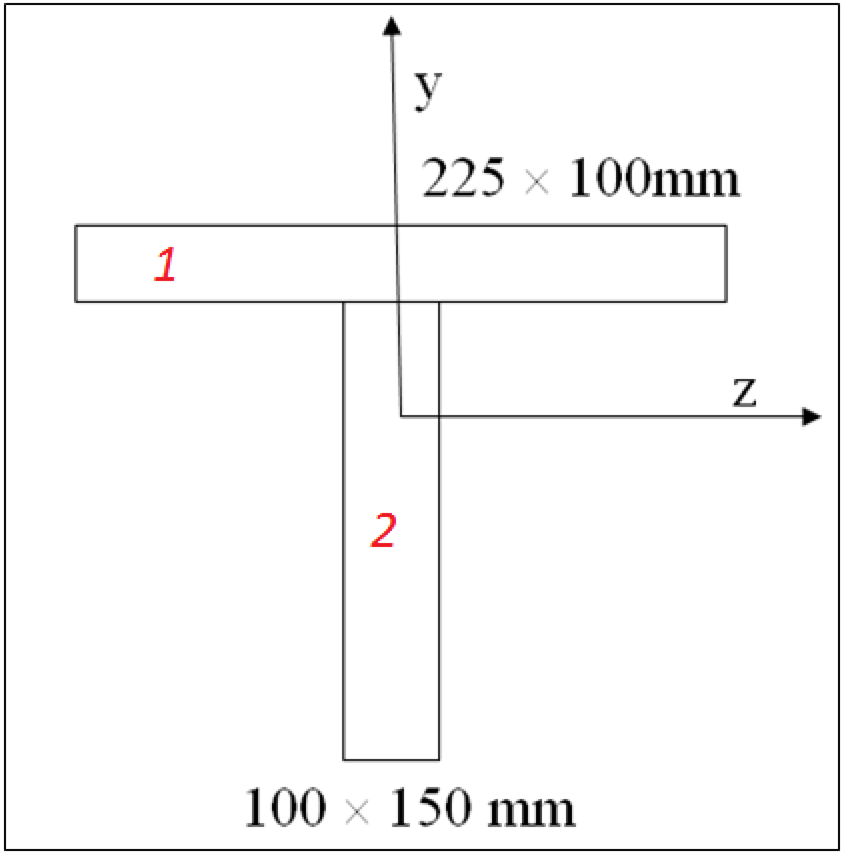
$$\boxed{(I_{ZZ})_1 = 75\times 10^6 mm^4}$$

$$(I_{ZZ})_2 = \frac{1}{12}\times 100\times(150)^3 + 100\times 150\times(75)^2$$

$$\boxed{(I_{ZZ})_2 = 1.125\times 10^8 mm^4}$$

$$I_{ZZ} = (I_{ZZ})_1 + (I_{ZZ})_2$$

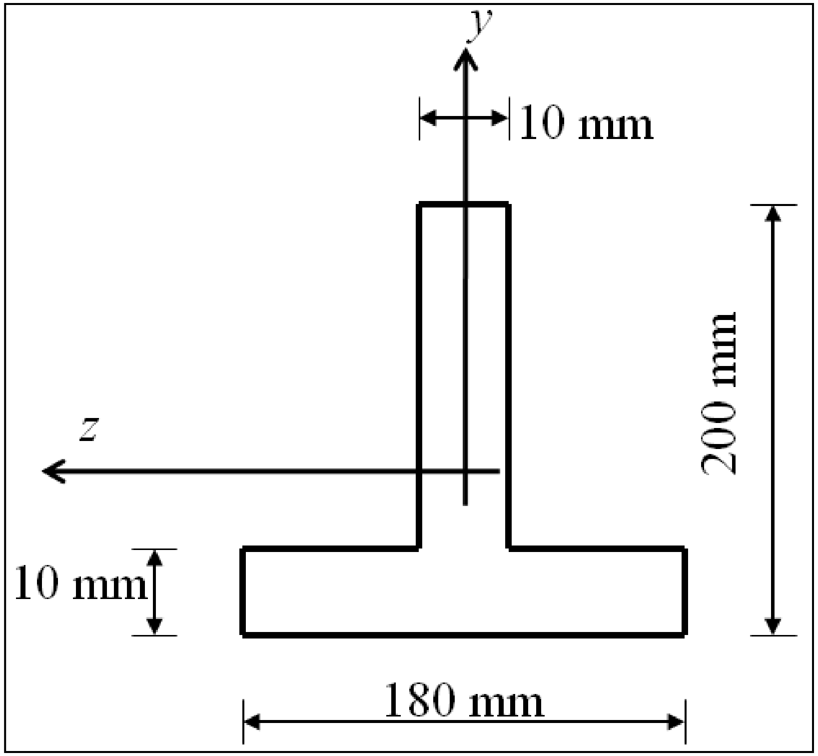
$$\boxed{I_{ZZ} = 1.875\times 10^8 mm^4}$$



Stresses due to Bending

Problem:

Calculate the moment of inertia for the beam cross section illustrated.

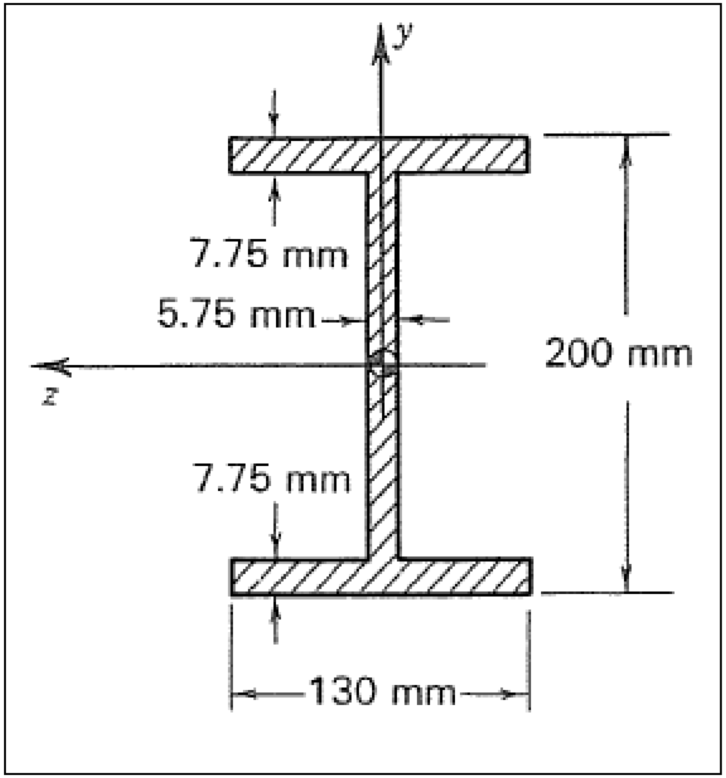


Ans: 14.97×10^{06} mm^4

Stresses due to Bending

Problem:

Calculate the moment of inertia for the beam cross section illustrated.

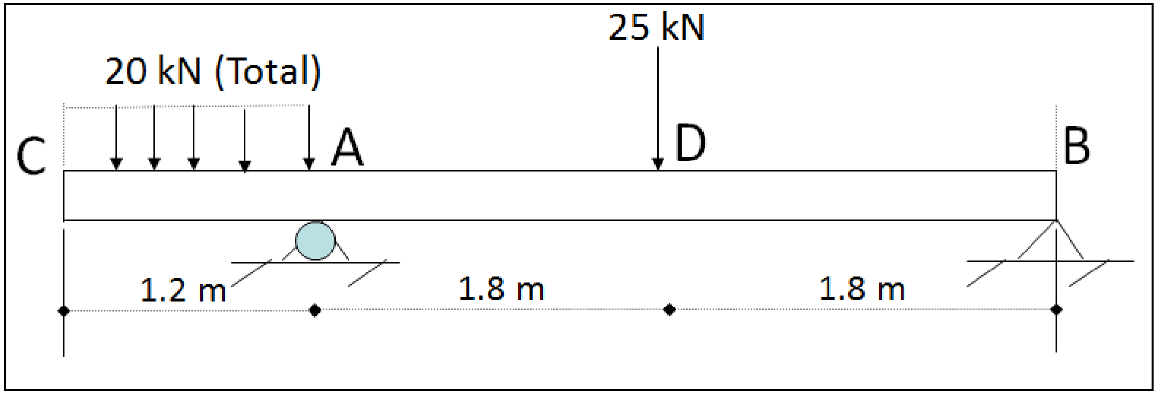


Ans: $21.64 \times 10^{06}\ mm^4$

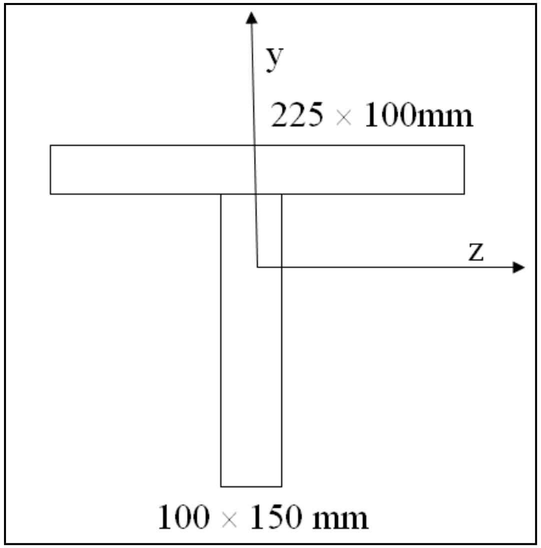
Stresses due to Bending

Problem:

A simply supported beam with over hang is loaded as shown in fig. cal the maximum bending stresses in the beam



Loading Diagram



Cross-section of the beam

Stresses due to Bending

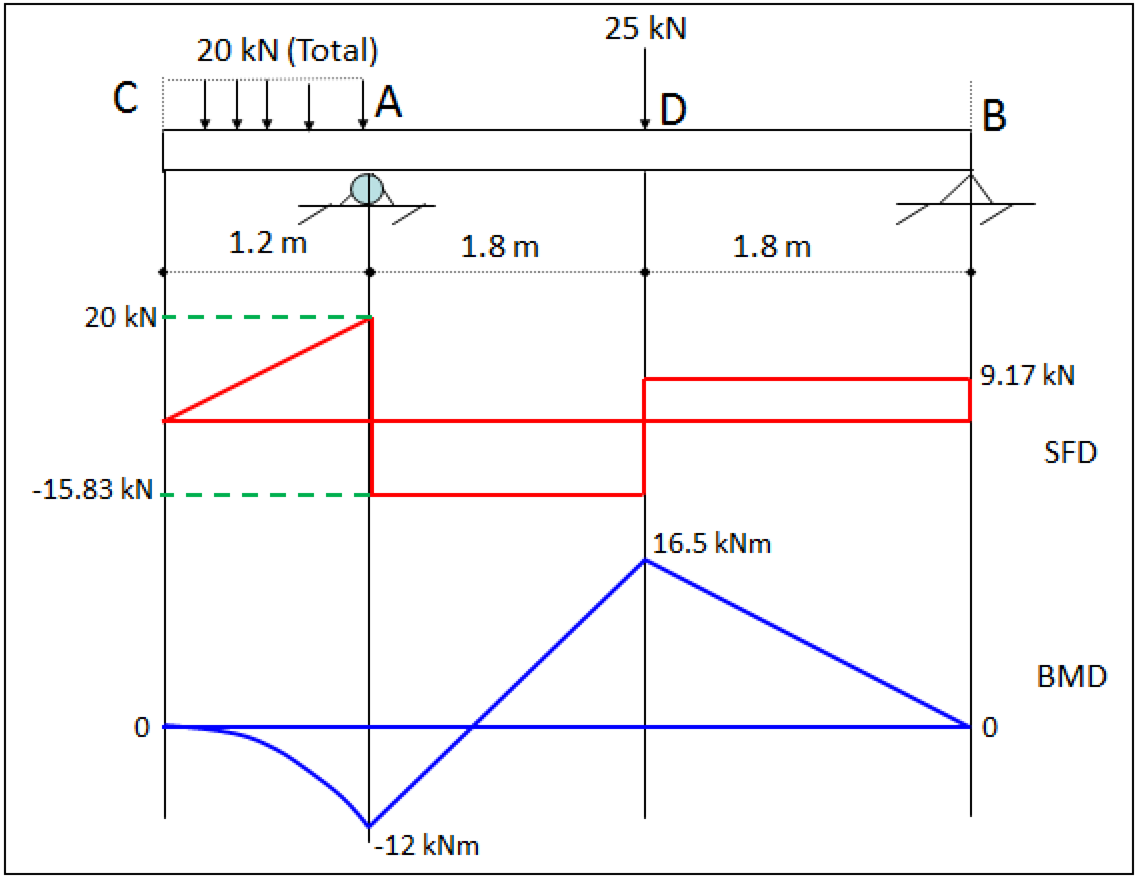
Solution

From chapter 3

$R_A = 35.83$ kN

$R_B = 9.17$ kN

Maximum bending moment is 16.5 kNm. Therefore beam should be designed based on 16.5 kNm



Stresses due to Bending

Solution

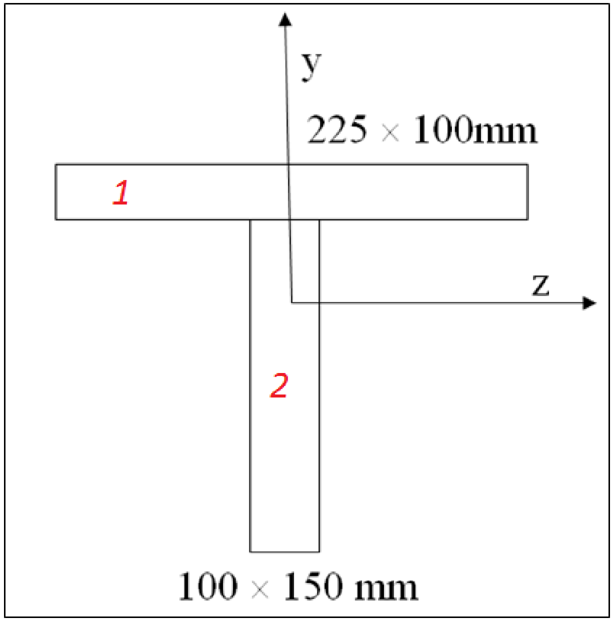
$$I_{ZZ} = 1.875 \times 10^8 \, mm^4$$

$$\frac{(\sigma_b)_t}{\bar{y}} = \frac{M_b}{I_{ZZ}} \quad \Rightarrow \quad (\sigma_b)_t = \frac{16.5 \times 10^6}{1.875 \times 10^8} \times 150$$

$$(\sigma_b)_t = 13.2 MPa$$

$$\frac{(\sigma_b)_c}{(250 - \bar{y})} = \frac{M_b}{I_{ZZ}} \quad \Rightarrow \quad (\sigma_b)_c = \frac{16.5 \times 10^6}{1.875 \times 10^8} \times 100$$

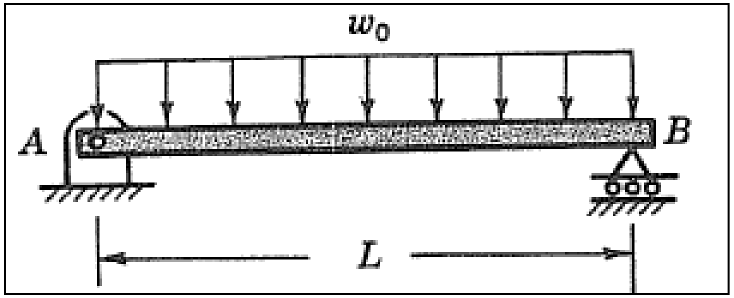
$$(\sigma_b)_c = 8.8 MPa$$



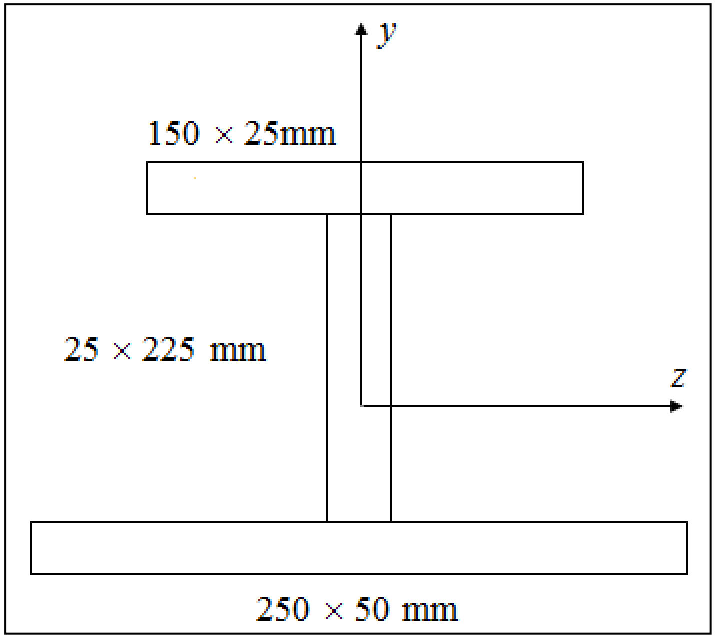
Stresses due to Bending

Problem:

A cast iron beam of I section as shown in fig is supported over a span of 5m. If the permissible stresses are 100 Mpa in compression and 25 Mpa in tension, what UDL will the beam carry safely?



Loading Diagram



Cross-section of the beam

Stresses due to Bending

Solution

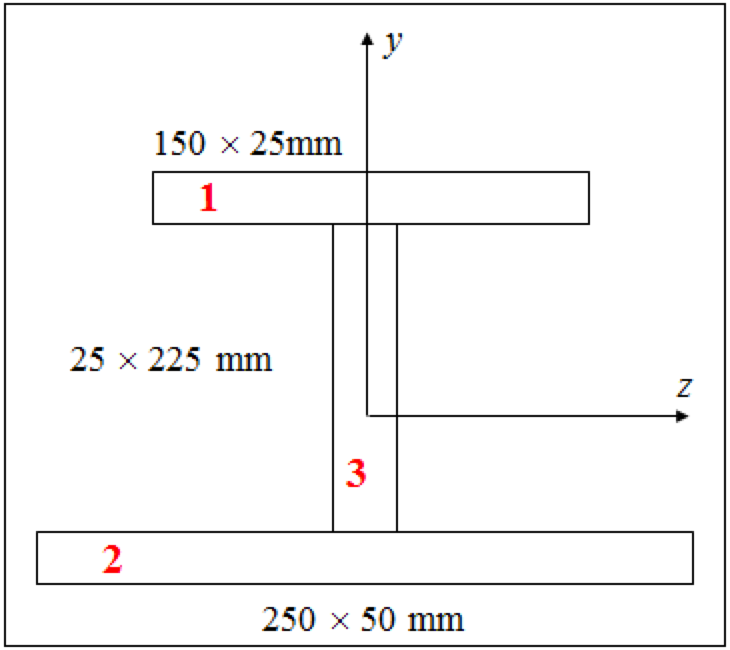
$$\bar{y}_1 = 287.5mm \quad \bar{y}_2 = 25mm \quad \bar{y}_3 = 162.5mm$$

$$\bar{y} = 105.36mm$$

$$I_{ZZ} = 2.5 \times 10^8 mm^4$$

$$\frac{(\sigma_b)_t}{\bar{y}} = \frac{M_b}{I_{ZZ}} \Rightarrow M_b = 59.32 \times 10^6 Nmm$$

$$\frac{(\sigma_b)_c}{(300 - \bar{y})} = \frac{M_b}{I_{ZZ}} \Rightarrow M_b = 128.44 \times 10^6 Nmm$$



For safe design, calculation will be against the minimum value of bending moment

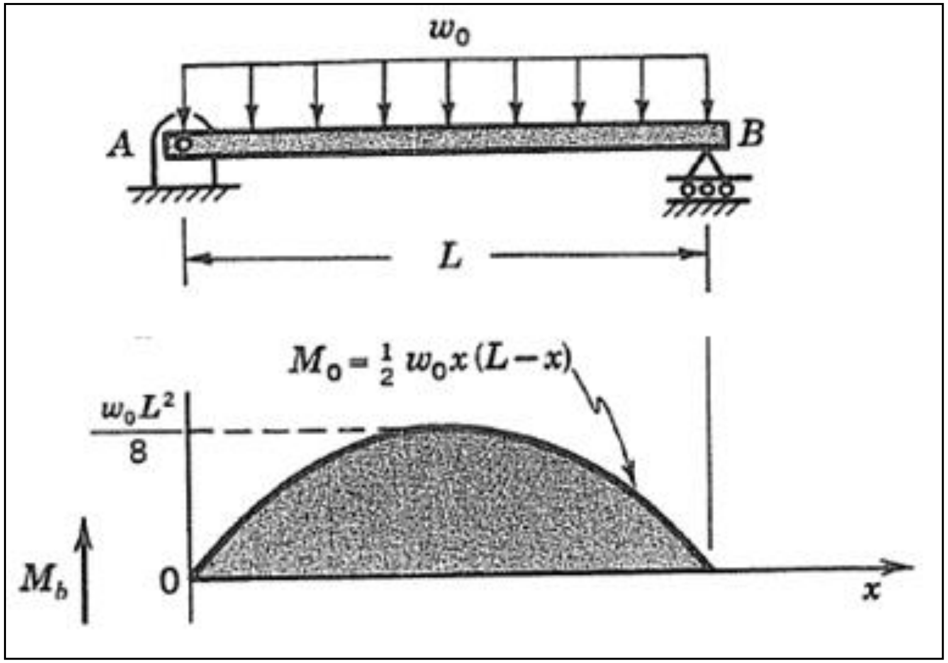
Stresses due to Bending

Solution

From chapter 3

Maximum bending moment for given loading diagram will act at center of the beam

$$M_b = \frac{w_o L^2}{8}$$



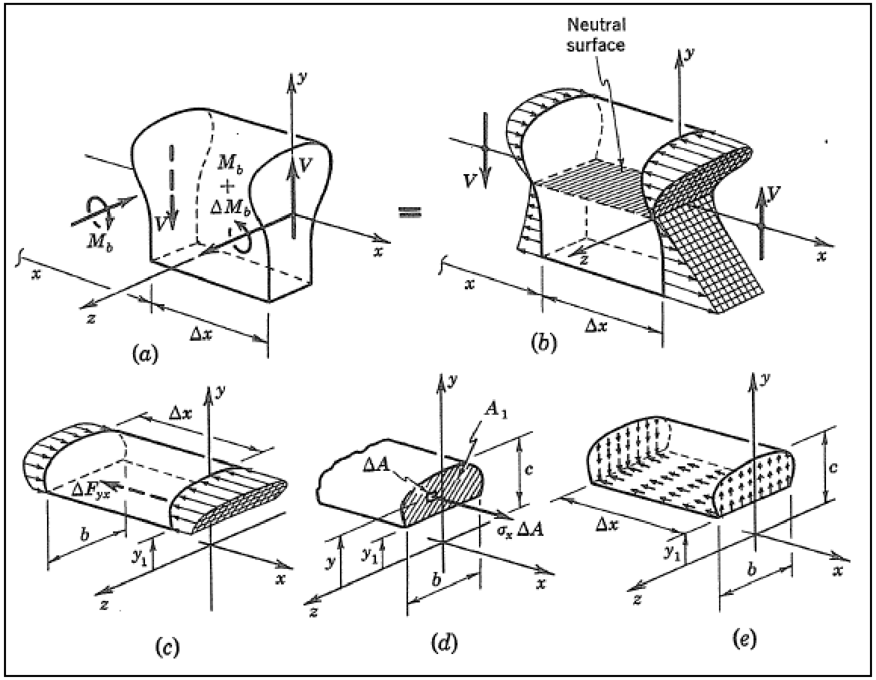
Permissible bending moment (M_b) will be = 59.32 x 10^6 Nmm

$$w_o = 18.98 \text{N/mm} \quad \text{OR} \quad w_o = 18.989 \text{kN/m}$$

Stresses due to Bending

Calculation of Shear Stress in a Symmetrical Beam from Equilibrium of a Segment of a Beam

- Constant shear force i.e. no external transverse load acting on the element.
- ΔM_b is variation of BM with x.
- *Fig. b* : Due to increase ΔM_b over length Δx, bending stresses acting on +ve x face of the beam element will be somewhat larger than those on the –ve x face.
- *Fig c*: equilibrium of segment of beam , by isolating part above plane $y=y_1$.



Stresses due to Bending

Calculation of Shear Stress in a Symmetrical Beam from Equilibrium of a Segment of a Beam

- Due to unbalance of bending stresses on the ends of this segment, ΔF_{yx} act on –ve y face to maintain force balance in the x direction.

$$\Sigma F_x = \left[\int_{A_1} \sigma_x dA\right]_{x+\Delta x} - \Delta F_{yx} - \left[\int_{A_1} \sigma_x dA\right]_x = 0$$

Where the integrals are to be taken over shaded area A_1 i.e. $y = y_1$ to $y = c$

$$\Delta F_{yx} = -\int_{A_1} \frac{(M_b + \Delta M_b)\, y}{I_{zz}} dA + \int_{A_1} \frac{M_b y}{I_{zz}} dA = -\frac{\Delta M_b}{I_{zz}} \int_{A_1} y dA$$

Dividing both sides by Δx and taking the limit

$$\frac{dF_{yx}}{dx} = \lim_{\Delta x \to 0} \frac{\Delta F_{yx}}{\Delta x} = -\frac{dM_b}{dx} \frac{1}{I_{zz}} \int_{A_1} y dA$$

Stresses due to Bending

Calculation of Shear Stress in a Symmetrical Beam from Equilibrium of a Segment of a Beam

We know that rate of change of BM is nothing but shear force i.e.

$$\frac{dM_b}{dx} = -V$$

Substitute above equation in previous one

$$\frac{dF_{yx}}{dx} = \frac{V}{I_{zz}} \int_{A_1} y dA$$

We may use following abbreviations for above equation

$$q_{yx} = \frac{dF_{yx}}{dx} \quad \text{and} \quad Q = \int_{A_1} y dA$$

Q is simply the first moment of shaded area A_1 and q_{yx} is shear flow i.e shear force per unit length.

Stresses due to Bending

Calculation of Shear Stress in a Symmetrical Beam from Equilibrium of a Segment of a Beam

$$q_{yx} = \frac{VQ}{I_{zz}}$$

Suppose width of beam is *b*, then shear stress τ_{yx} or τ_{xy} is given by

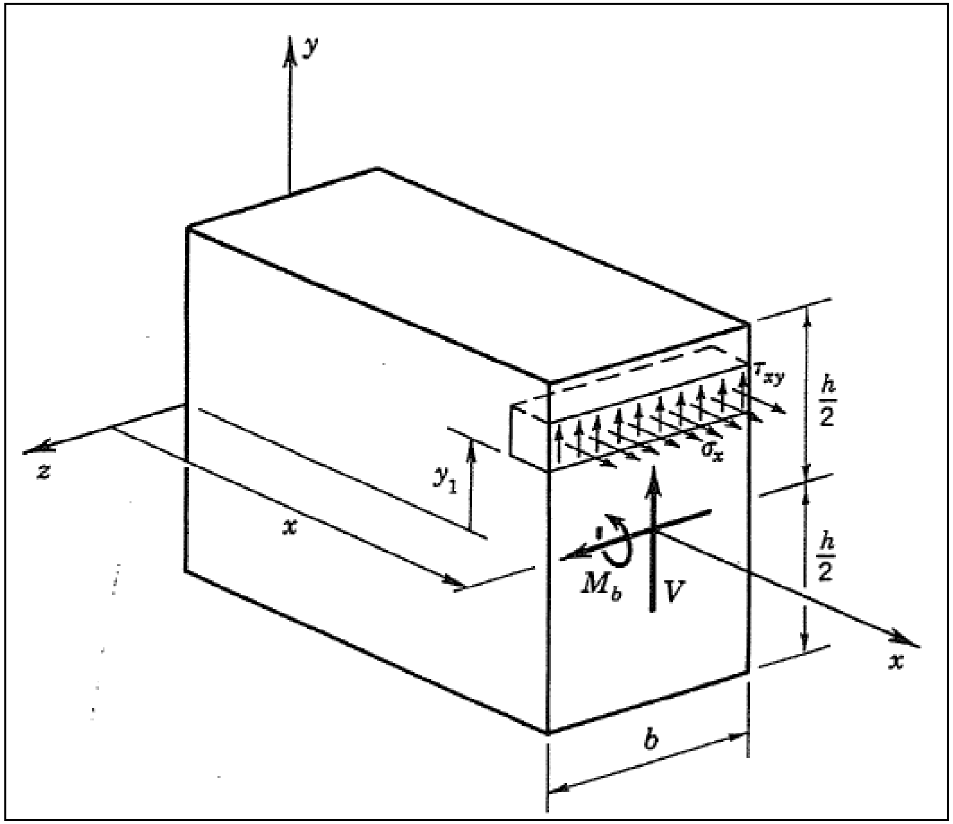
$$\tau_{yx} = \tau_{xy} = \frac{q_{yx}}{b} = \frac{VQ}{bI_{zz}}$$

Stresses due to Bending

Shear Stress Distribution in Rectangular Beams

Consider equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$$



Our assumption says that shear force; therefore shear stress also is independent of *x*

$$-\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial \sigma_x}{\partial x}$$

$$-\frac{\partial \tau_{xy}}{\partial y} = \frac{\partial}{\partial x}\left(-\frac{M_b y}{I_{zz}}\right) = \frac{Vy}{I_{zz}}$$

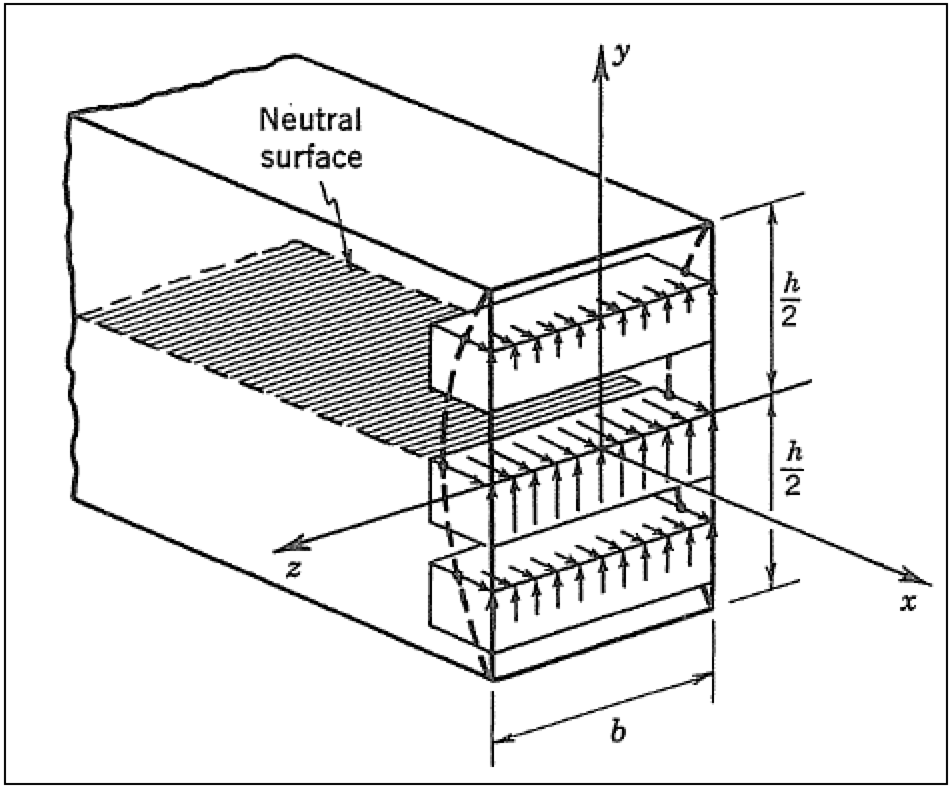
Stresses due to Bending

Shear Stress Distribution in Rectangular Beams

$$-\int_{y_1}^{h/2} \frac{\tau_{xy}}{\partial y} dy = \frac{V}{I_{zz}} \int_{y_1}^{h/2} y dy$$

$$-\left[\tau_{xy}\right]_{y_1}^{h/2} = \frac{V}{I_{zz}} \left[\frac{y^2}{2}\right]_{y_1}^{h/2}$$

$$\tau_{xy} = \frac{V}{2I_{zz}} \left[\left(\frac{h}{2}\right)^2 - y_1^2\right]$$

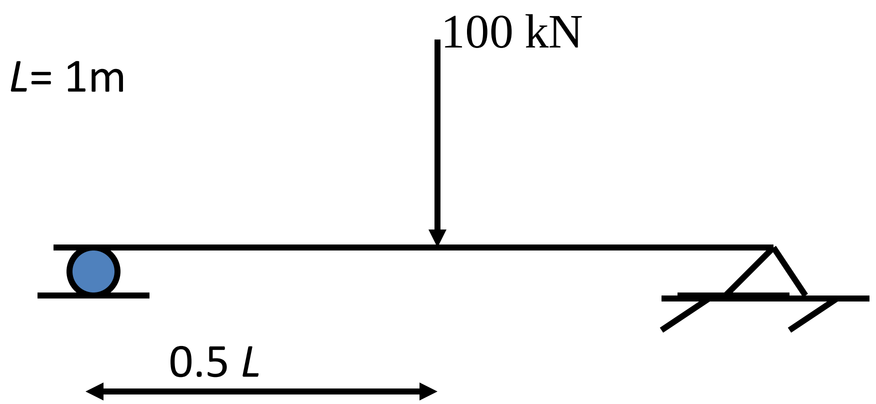


Shear stress is maximum at neutral surface and it has parabolic variation across the cross section of the beam.

Stresses due to Bending

Problem:

Sketch the shear stress distribution diagram for the given I section where the shear force is maximum along the length of the beam. Also determine the ratio of max bending stress to max shear stress



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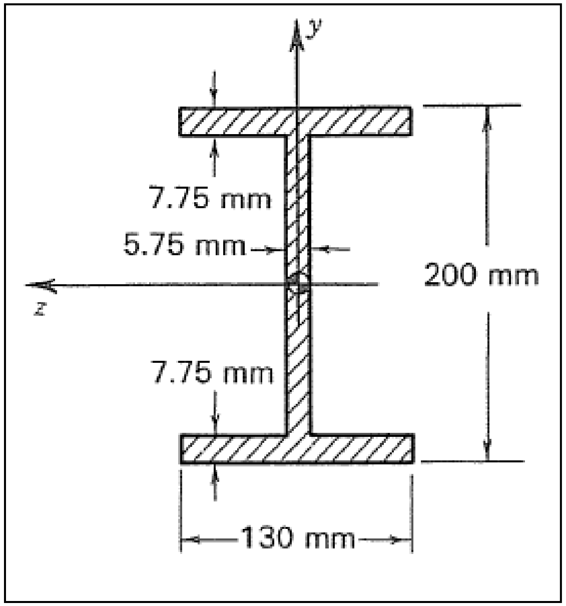


Cross-section of the beam

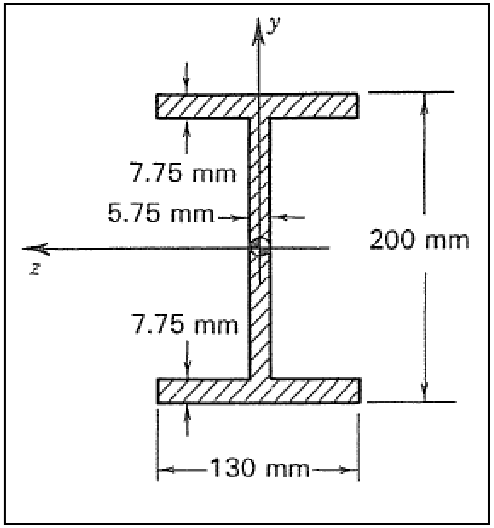
Stresses due to Bending

Solution:

Consider Loading diagram. Refer ch. 3 to determine maximum bending moment and maximum shear force.

Maximum shear force, V = 50kN

Maximum bending moment, M_b = 25kNm

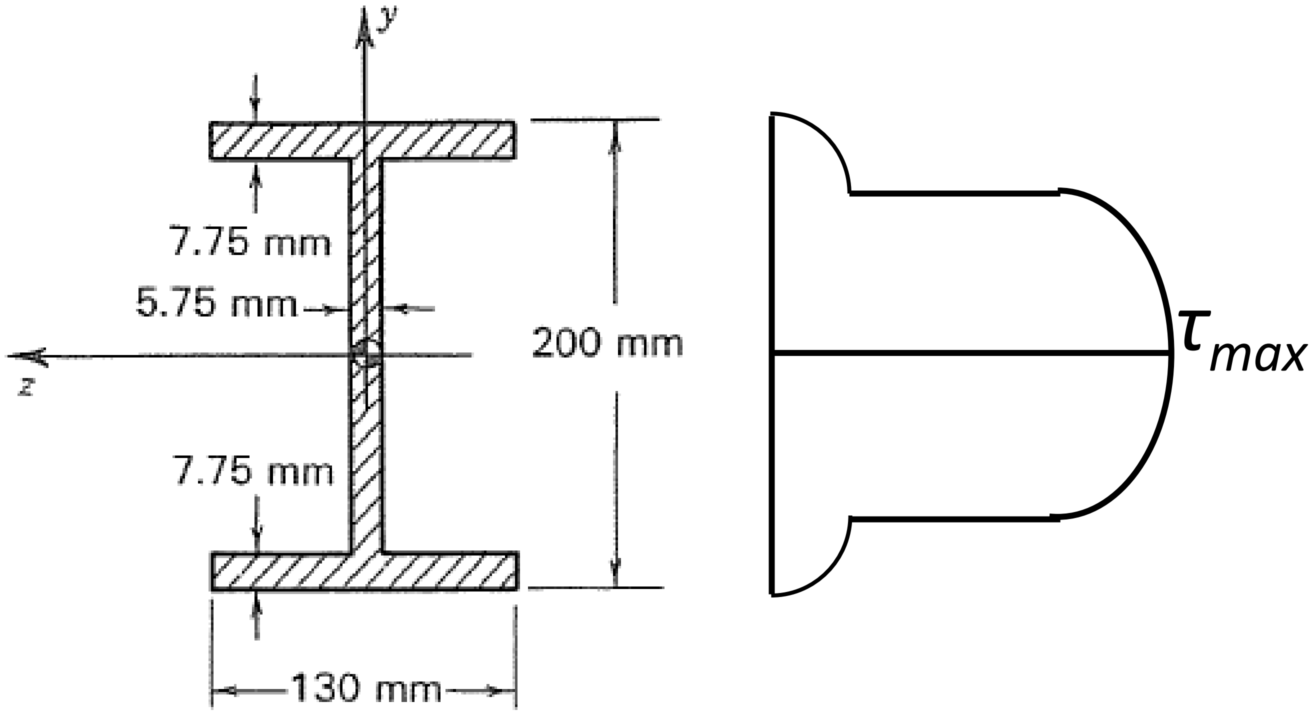


Moment of inertia for given I section is (refer earlier slides)

I_{zz}: 21.64 x 10^{06} mm^4

Stresses due to Bending

Solution:

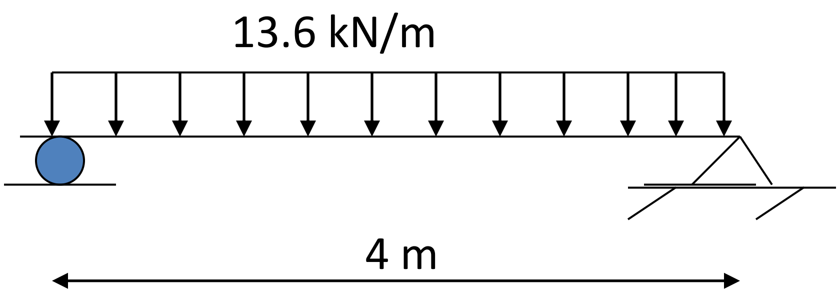


Shear stress distribution Cross-section of the beam

Stresses due to Bending

Problem:

Sketch the shear stress distribution diagram for the given T section where the shear force is maximum along the length of the beam. Also determine the ratio of max bending stress to max shear stress



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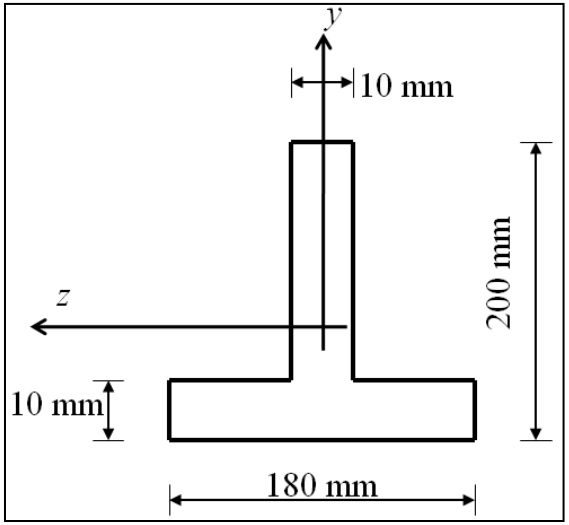


Cross-section of the beam

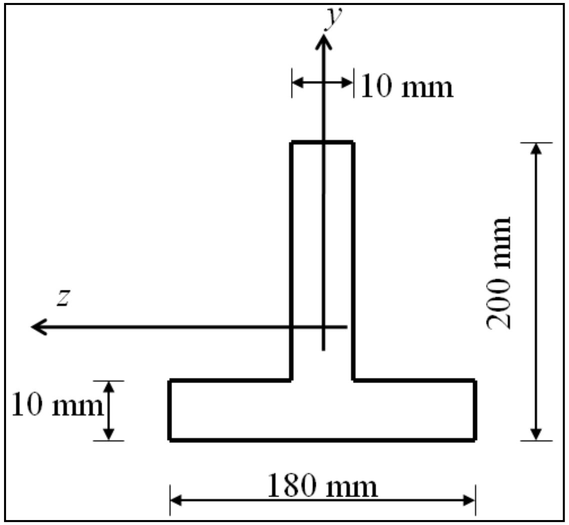
Stresses due to Bending

Solution:

Consider Loading diagram. Refer ch. 3 to determine maximum bending moment and maximum shear force.

Maximum shear force, V = 27.2kN

Maximum bending moment, M_b = 27.2kNm

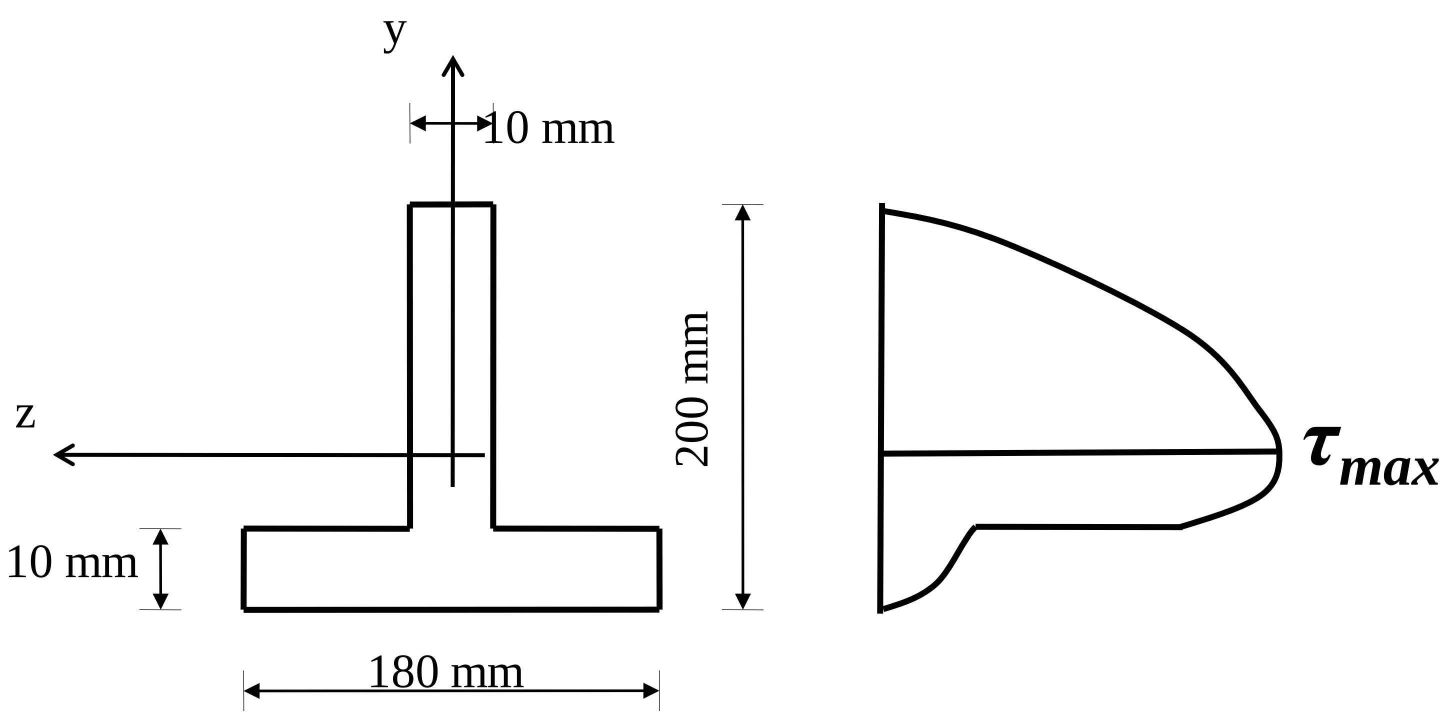


Moment of inertia for given T section is (refer earlier slides)

I_{zz}: 14.97×10^{06} mm^4

Stresses due to Bending

Solution:

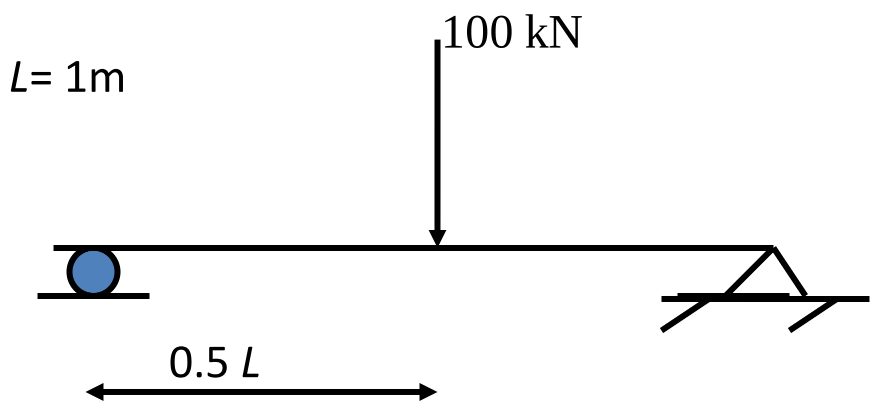


Shear stress distribution Cross-section of the beam

Stresses due to Bending

Problem:

Sketch the shear stress distribution diagram for the given T section where the shear force is maximum along the length of the beam. Also determine the ratio of max bending stress to max shear stress



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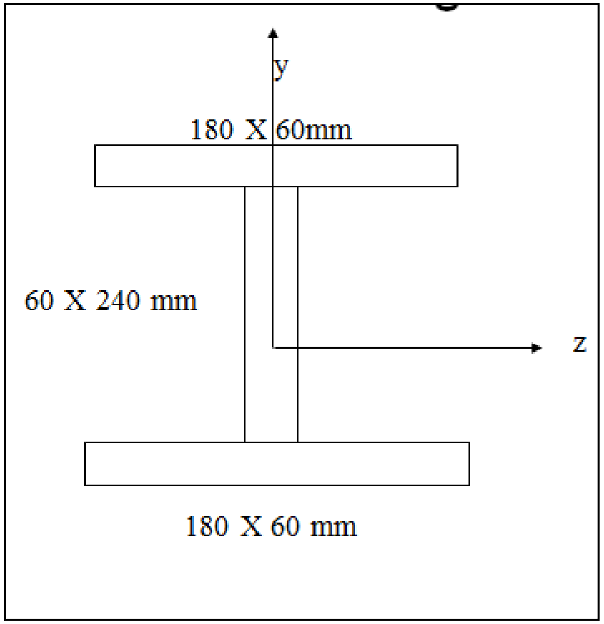
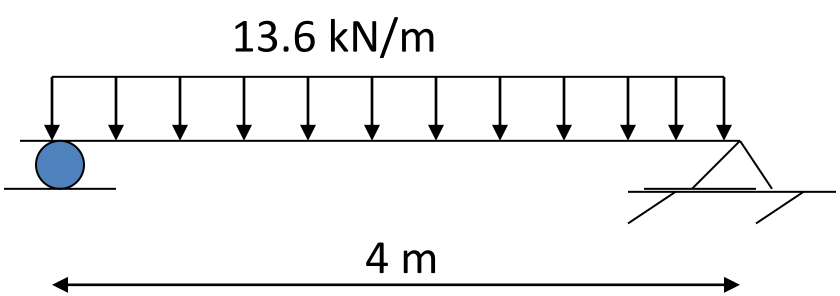


Cross-section of the beam

Problem:

Sketch the shear stress distribution diagram for the given T section where the shear force is maximum along the length of the beam. Also determine the ratio of max bending stress to max shear stress



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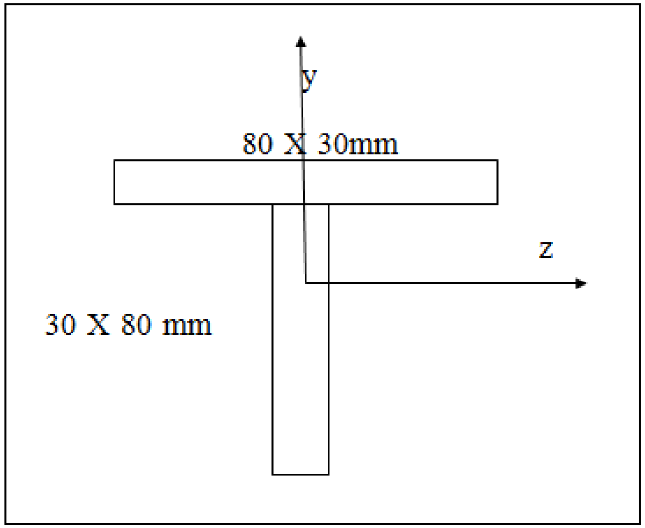
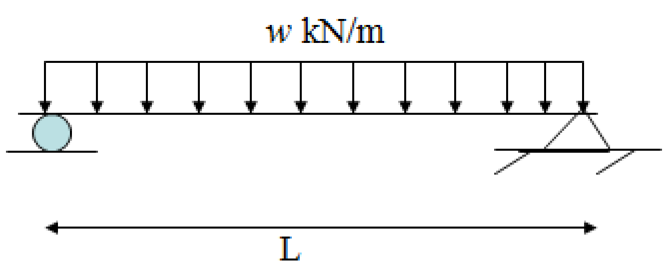


Cross-section of the beam

Stresses due to Bending

Problem:

Determine the ratio of max bending stress to max shear stress



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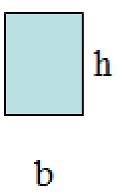

Cross-section of the beam

References

1. Introduction to Mechanics of Solids by S. H. Crandall et al (In SI units), McGraw-Hill